Name of College - S.S. College, Jehanabad

Deptt - Mathematics

Date - 21-05-2020

Time - 10 A.M to 11 A.M

Topic - Beta and Gamma Function (Contd.)

(Improper integral Real Analysis)

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1. Show that-

$$\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi} \qquad 0<n<1$$

Solution → We have

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}$$

Therefore

$$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad \text{——} \quad (1)$$

Further, we have

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = B(m,n)$$

Therefore, $\frac{\cancel{\Gamma m \, \Gamma n}}{\Gamma m+n}$ = Putting $m+n=1$

So that $m = 1-n$

and we get-

$$\frac{\Gamma 1-n \, \Gamma n}{\Gamma 1} = B(1-n, n)$$

Therefore $\Gamma n \, \Gamma 1-n = \int_0^\infty \frac{x^{n-1}}{1+x} dx$

From (1)

But $\int_0^\infty \frac{x^{a-1}}{1+x} dx = \frac{\pi}{\sin a\pi}$

Therefore, $\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$

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Show that-

$$\Gamma n \,\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma 2n$$

This is also known as Duplication formula :→

Solution : We know that-

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \,\Gamma n}{2\,\Gamma(m+n)} \quad \text{—— (1)}$$

Now putting $n = \frac{1}{2}$ or $2n-1 = 0$

Therefore, $$\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta = \frac{\Gamma\frac{1}{2}\,\Gamma m}{2\,\Gamma\left(m+\frac{1}{2}\right)} \quad \text{—— (2)}$$

Now putting $m = n$ in (1), we get-

$$\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

$$\Rightarrow \frac{1}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

Now putting $2\theta = z \Rightarrow \theta = z/2$

When $\theta = 0,\ z = 0$

and when $\theta = \pi/2,\ z = \pi$

We get-

$$\frac{(\Gamma m)^2}{2\,\Gamma 2m} = \frac{1}{2^{2m}} \int_0^{\pi} (\sin z)^{2m-1} dz$$

But we know that-

$$\int_0^{\pi} \sin^n x \, dx = 2\int_0^{\pi/2} \sin^n x \, dx$$

Therefore,

$$\frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} z\, dz = \frac{(\Gamma m)^2}{2\,\Gamma 2m}$$

or $$\frac{1}{2^{2m-1}}\;\frac{\Gamma(\tfrac{1}{2})\,\Gamma m}{2\,\Gamma m+\tfrac{1}{2}} = \frac{(\Gamma m)^2}{2\,\Gamma 2m}.$$

Therefore,

$$\Gamma m\;\Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m$$

or, $$\Gamma n\;\Gamma n+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n)$$ Proved

Find the value of

$\Gamma 1/n\;\;\Gamma 2/n\;\;\Gamma 3/n\;\cdots\;\Gamma\frac{n-1}{n}$; n being +ve integer.

Solution :→ Suppose

$$P = \Gamma 1/n\;\;\Gamma 2/n\;\;\Gamma 3/n\;\cdots\;\Gamma\frac{n-1}{n}$$

But $\frac{n-1}{n} = 1-\frac{1}{n}$ and $\frac{1}{n} = 1-\frac{n-1}{n}$

Now ~~Writi~~ Writing in the reverse order, we get

$$P = \Gamma(1-\tfrac{1}{n})\,\Gamma(1-\tfrac{2}{n})\cdots\Gamma\left(1-\frac{n-1}{n}\right)$$

Multiplying P, we get

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$$P^2 = \left[\Gamma\tfrac{1}{n}\,\Gamma\left(1-\tfrac{1}{n}\right)\right]\left[\Gamma\tfrac{2}{n}\cdot\Gamma\left(1-\tfrac{2}{n}\right)\right]\cdots\left[\Gamma\tfrac{n-1}{n}\cdot\Gamma\left(1-\tfrac{n-1}{n}\right)\right]$$

But we know that

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Therefore,

$$P^2 = \frac{\pi}{\sin\frac{\pi}{n}} \times \frac{\pi}{\sin\frac{2\pi}{n}} \times \frac{\pi}{\sin\frac{3\pi}{n}} \cdots \frac{\pi}{\sin(n-1)\frac{\pi}{n}} \quad \text{--- (1)}$$

Now we have to find the value of the expression on the right. We know that

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\left(\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\cdots\sin\left(\theta+\frac{n-1}{n}\pi\right)\right)$$

Taking limit as $\theta \to 0$

$$\lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = \lim_{\theta\to 0} 2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\cdots\sin\left(\theta+\frac{n-1}{n}\pi\right) \quad \text{--- (2)}$$

$$\text{But } \lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{\sin n\theta}{n\theta}\cdot\frac{n\theta}{\frac{\sin\theta}{\theta}\times\theta}$$

$$= 1\cdot 1\cdot n$$

Therefore, from (2)

$$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{n-1}{n}\pi$$

Thus from (1), we get

$$P^2 = \frac{\pi^{n-1}}{\frac{n}{2^{n-1}}} = \frac{(2\pi)^{n-1}}{n}$$

$$\text{Therefore } P = \Gamma\tfrac{1}{n}\,\Gamma\tfrac{2}{n}\,\Gamma\tfrac{3}{n}\cdots\Gamma\tfrac{n-1}{n}$$

$$= \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

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Prove that

$$\Gamma n \; \Gamma\frac{1-n}{2} = \frac{\sqrt{\pi}\;\Gamma\frac{n}{2}}{2^{1-n}\cos\frac{n\pi}{2}} \qquad 0<n<1$$

Solution :→ we know that

$$\Gamma m \; \Gamma 1-m = \frac{\pi}{\sin m\pi} \quad \text{—— (i)}$$

and $$\Gamma m \; \Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\;\Gamma 2m \quad \text{—— (ii)}$$

Putting $m = \frac{n+1}{2}$ in (i),

we get $$\Gamma(n+\tfrac{1}{2})\;\Gamma\frac{1-n}{2} = \frac{\pi}{\cos\frac{n\pi}{2}} \quad \text{—— (iii)}$$

Now writing $2m = n$ in (ii)

we have

$$\Gamma n/2 \;\; \Gamma\frac{n+1}{2} = \frac{\sqrt{\pi}}{n+}\;\Gamma n \quad \text{—— (iv)}$$

Now Divide (iii) by (iv)

$$\frac{\Gamma\frac{n+1}{2}\;\Gamma\frac{1-n}{2}}{\Gamma n/2 \;\; \Gamma\frac{n+1}{2}} = \frac{\dfrac{\pi}{\cos\frac{n\pi}{2}}}{\dfrac{\sqrt{\pi}\;\Gamma n}{2^{n-1}}}$$

$$\Rightarrow \frac{\Gamma\frac{1-n}{2}}{\Gamma(n/2)} = \frac{\sqrt{\pi}}{2^{1-n}\;\Gamma n \cos(\frac{n\pi}{2})}$$

$$\Rightarrow \Gamma n \; \Gamma\frac{1-n}{2} = \frac{\sqrt{\pi}\;\Gamma n/2}{2^{1-n}\cos(\frac{n\pi}{2})}$$